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TECHNICAL NOTES

Non-Darcy convection from horizontal impermeable surfaces in saturated porous media

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INTRODUCTION

IN RECENT years heat transfer in saturated porous media has received considerable attention because of its important applications in geophysics and energy related engineering problems. In most of the previous studies [1–6], either for natural or mixed convection, the boundary-layer formulation of Darcy's law and the energy equation were used. However, the non-Darcy flow situation which may prevail in some of the above applications has not received much attention. Recently, Plumb and Huenefeld [7], and Vasantha *et al.* [8] have reported the non-Darcy natural convection for different configurations in saturated porous media by employing the Ergun model [9].

In this paper, the same non-Darcy flow model is used and steady non-Darcy convection, in the form of natural, mixed and forced convection, is considered for a heated horizontal surface embedded in a saturated porous medium. Under the assumed conditions, a similarity solution exists for the case of constant surface heat flux.

ANALYSIS

Consider two-dimensional non-Darcy convection over a horizontal impermeable surface embedded in a saturated porous medium. In the mathematical formulation of the problem, we assume (i) constant fluid and medium (isotropic) properties and (ii) local thermodynamic equilibrium between fluid and solid phases. Under these assumptions, the governing equations are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u+\frac{K'}{\mu}u^2=-\frac{K}{\mu}\frac{\partial p}{\partial x} \tag{2}$$

$$v+\frac{K'}{\mu}v^2=-\frac{K}{\mu}\left(\frac{\partial p}{\partial y}+\rho g\right) \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right). \tag{4}$$

The boundary conditions are

$$y=0,\ T_w=T_\infty+Ax^\lambda,\ v=0$$

$$y\to\infty,\ T=T_\infty,\ u=0\quad \text{(natural convection)} \tag{5}$$

$$=U_\infty=Bx^m\quad \text{(mixed convection)}. \tag{6}$$

As can be seen, the major difference between the Ergun and the Darcy models is that the former takes into account the inertial effect. A detailed discussion of the limitation of Darcy's law and non-Darcy flow models is presented by Bear [10].

When the Boussinesq and boundary-layer approximations are invoked, the governing equations in terms of stream function, ψ, are reduced to

$$\frac{\partial^2\psi}{\partial y^2}+\frac{K'}{\nu}\frac{\partial}{\partial y}\left[\left(\frac{\partial\psi}{\partial y}\right)^2\right]=-\frac{Kg\beta}{\nu}\frac{\partial T}{\partial x} \tag{7}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}. \tag{8}$$

With the properly chosen similarity variables, equations (7) and (8) can be transformed to a set of ordinary differential equations.

Natural convection

The suitable similarity variables for solving the non-Darcy natural convection problem are those introduced by Cheng and Chang [3]

$$\eta=(Ra)^{1/3}\frac{y}{x} \tag{9}$$

$$\psi=\alpha(Ra)^{1/3}f(\eta) \tag{10}$$

$$\theta=(T-T_\infty)/(T_w-T_\infty). \tag{11}$$

After transformation, the resulting equations are

$$f''+\frac{K'\alpha}{\nu}\left[\frac{Kg\beta A}{\nu\alpha}\right]^{2/3}x^{(2\lambda-1)/3}[(f')^2]'+\lambda\theta+\frac{\lambda-2}{3}\eta\theta'=0 \tag{12}$$

$$\theta''-\lambda f'\theta+\left(\frac{\lambda+1}{3}\right)f\theta'=0. \tag{13}$$

Obviously, the equations will be independent of x, if $\lambda=1/2$,

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NOMENCLATURE

- A constant defined in equation (5)
- B constant defined in equation (6)
- f dimensionless stream function
- G^* parameter of the inertial effect on natural convection, $(K'\alpha/\nu)(Kg\beta A/\nu\alpha)^{2/3}$
- h local heat transfer coefficient
- k effective thermal conductivity of the saturated porous medium
- K permeability
- K' inertial coefficient of the Ergun equation
- Nu local Nusselt number, hx/k
- P pressure
- Pe Peclet number, $U_\infty x/\alpha$
- R^* parameter of the inertial effect on mixed convection, $K'U_\infty/\nu$
- Ra modified Rayleigh number, $Kg\beta(T_w - T_\infty)x/\nu\alpha$
- T temperature
- T_w surface temperature
- T_∞ ambient temperature or free stream temperature
- u, v velocity components in x- and y-direction
- U_∞ free stream velocity
- x, y Cartesian coordinates.

Greek symbols
- α effective thermal diffusivity of saturated porous medium
- β coefficient of thermal expansion
- η independent similarity variable
- θ dimensionless temperature
- μ dynamic viscosity
- ν kinematic viscosity
- ρ fluid density
- ψ stream function.

Subscripts
- nc natural convection
- mx mixed convection.

that is, the surface temperature varies with $x^{1/2}$. Thus, equations (12) and (13) can be further reduced to

$$f'' + G^*[(f')^2]' + \frac{\theta}{2} - \frac{\eta}{2}\theta' = 0 \quad (14)$$

$$\theta'' - \tfrac{1}{2}(f'\theta - f\theta') = 0 \quad (15)$$

where

$$G^* = \frac{K'\alpha}{\nu}\left[\frac{Kg\beta A}{\nu\alpha}\right]^{2/3}$$

and the boundary conditions are

$$\eta = 0, \quad \theta = 1, \quad f = 0 \quad (16)$$

$$\eta \to \infty, \quad \theta = 0, \quad f' = 0. \quad (17)$$

Mixed convection

The similarity variables applicable for non-Darcy mixed convection have also been introduced by Cheng [5]

$$\eta = \left(\frac{U_\infty x}{\alpha}\right)^{1/2}\frac{y}{x} \quad (18)$$

$$\psi = (\alpha U_\infty x)^{1/2} f(\eta). \quad (19)$$

Equations (7) and (8) are transformed to

$$f'' + \frac{K'B}{\nu}x^m[(f')^2]' = -\frac{Kg\beta A}{\nu B}\left[\frac{\alpha}{B}\right]^{1/2} x^{(2\lambda-1-3m)/2}\left[\frac{m-1}{2}\eta\theta' + \lambda\theta\right] \quad (20)$$

$$\theta'' = \lambda\theta f' - \frac{1+m}{2}f\theta'. \quad (21)$$

These will be independent of x if $m = 0$ and $\lambda = 1/2$, that is, a uniform flow over a horizontal surface where the surface temperature varies with $x^{1/2}$. Then, the above equations can be reduced to

$$f'' + R^*[(f')^2]' = \frac{Ra}{2Pe^{3/2}}(\eta\theta' - \theta) \quad (22)$$

$$\theta'' = \tfrac{1}{2}(\theta f' - f\theta') \quad (23)$$

where $R^* = K'U_\infty/\nu$, and the corresponding boundary conditions are

$$\eta = 0, \quad \theta = 1, \quad f = 0 \quad (24)$$

$$\eta \to \infty, \quad \theta = 0, \quad f' = 1. \quad (25)$$

Forced convection

It is noted that the governing equations for non-Darcy forced convection can be deduced from equations (22) and (23) by simply setting $Ra/(Pe)^{3/2} = 0$. Therefore

$$f'' + R^*[(f')^2]' = 0 \quad (26)$$

$$\theta'' = \tfrac{1}{2}[\theta f' - f\theta']. \quad (27)$$

RESULTS AND DISCUSSION

The transformed ordinary differential equations, with the corresponding boundary conditions, are solved by numerical integration using the Runge–Kutta method by the shooting technique with a systematic guessing of $\theta'(0)$ and $f'(0)$.

The resulting profiles of dimensionless velocity and temperature are shown in Figs. 1 and 2 for natural convection, and in Figs. 3 and 4 for mixed convection. Selected values of $-\theta'(0)$ and $f'(0)$ are also given in Table 1. It is interesting to note that the solutions of equations (26) and (27) turn out to be the same as those of Darcy forced convection. This

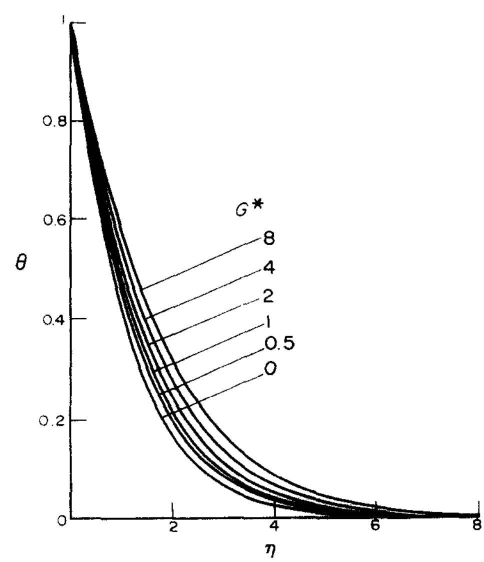

FIG. 1. Dimensionless temperature profile for non-Darcy natural convection.

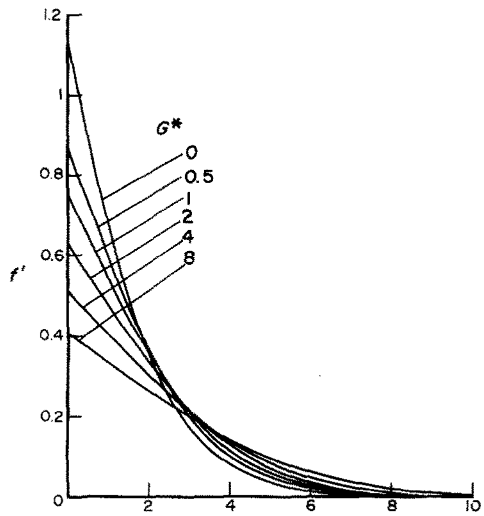

FIG. 2. Dimensionless velocity profile for non-Darcy natural convection.

implies that the non-Darcy term has little significance in forced convection, which is true because of the boundary-layer approximation used in the analysis.

The local surface heat flux for natural convection is given by

$$q_{nc} = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = kA^{4/3}\left[\frac{Kg\beta}{\nu\alpha}\right]^{1/3} \times x^{(4\lambda-2)/3}[-\theta'(0)] \quad (28a)$$

and for mixed convection

$$q_{mx} = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = kA\left(\frac{\beta}{\alpha}\right)^{1/2} \times x^{(2\lambda+m-1)/2}[-\theta'(0)]. \quad (28b)$$

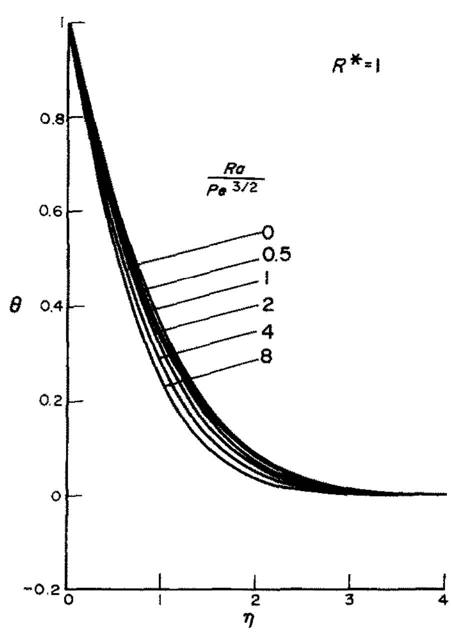

FIG. 3. Dimensionless temperature profile for non-Darcy mixed convection.

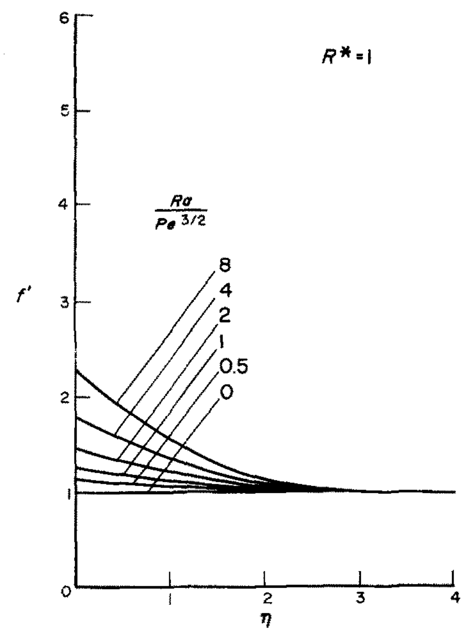

FIG. 4. Dimensionless velocity profile for non-Darcy mixed convection.

For $\lambda = 1/2$ and $m = 0$, these correspond to the case of constant surface heat flux.

The heat transfer coefficient in terms of the Nusselt number can be expressed as

$$\text{natural convection} \qquad \frac{Nu}{(Ra)^{1/3}} = -[\theta'(0)]_{nc} \quad (29)$$

$$\text{mixed convection} \qquad \frac{Nu}{(Pe)^{1/2}} = -[\theta'(0)]_{mx} \quad (30)$$

and

$$\text{forced convection} \qquad \frac{Nu}{(Pe)^{1/2}} = 0.8862. \quad (31)$$

Equation (30) is plotted in Fig. 5 as a function of $Ra/(Pe)^{3/2}$. The limiting cases of free convection and forced convection are also shown as asymptotes in the same figure. It is noted that the asymptotes of non-Darcy pure natural convection are not linear, which is different from the Darcy case. The corresponding free convection asymptotes can be obtained

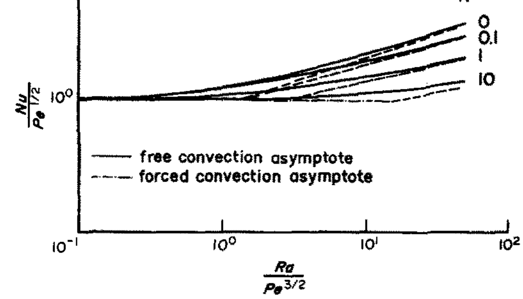

FIG. 5. Heat transfer results for non-Darcy mixed convection.

Table 1. Values of $-\theta'(0)$ and $f'(0)$ for non-Darcy mixed and natural convection

	$R^* = 0$		$R^* = 0.1$		$R^* = 1$		$R^* = 10$				
$Ra/Pe^{3/2}$	$-\theta'(0)$	$f'(0)$	$-\theta'(0)$	$f'(0)$	$-\theta'(0)$	$f'(0)$	$-\theta'(0)$	$f'(0)$	G^*	$-\theta'(0)$	$f'(0)$
0.1	0.9137	1.0865	0.9092	1.0720	0.8956	1.0291	0.8876	1.0042	0.1	0.7919	1.0557
0.5	1.0077	1.4015	0.9878	1.3306	0.9295	1.1362	0.8929	1.0207	0.5	0.7299	0.8683
1	1.1020	1.7474	1.0666	1.6082	0.9661	1.2547	0.8994	1.0409	1	0.6848	0.7508
2	1.2495	2.3479	1.1883	2.0729	1.0266	1.4569	0.9117	1.0795	2	0.6304	0.6266
4	1.4645	3.3543	1.3614	2.8077	1.1180	1.7788	0.9343	1.1511	4	0.5715	0.5084
8	1.7610	4.9990	1.5904	3.9125	1.2446	2.2595	0.9735	1.2773	8	0.5121	0.4042
10	1.8763	5.7191	1.6763	4.3657	1.2932	2.4551	0.9907	1.3337	10	0.4933	0.3740
20	2.3059	8.8010	1.9820	6.1500	1.4688	3.2145	1.0620	1.5740	13.57	0.4679	0.3355
30	2.6139	11.3978	2.1884	7.5060	1.5888	3.7829	1.1174	1.7688	15.87	0.4552	0.3171
40	2.8614	13.7193	2.3474	8.6340	1.6820	4.2528	1.1633	1.9357	29.24	0.4078	0.2534
50	3.0713	15.8529	2.4780	9.6144	1.7588	4.6582	1.2027	2.0834	46.42	0.3737	0.2124

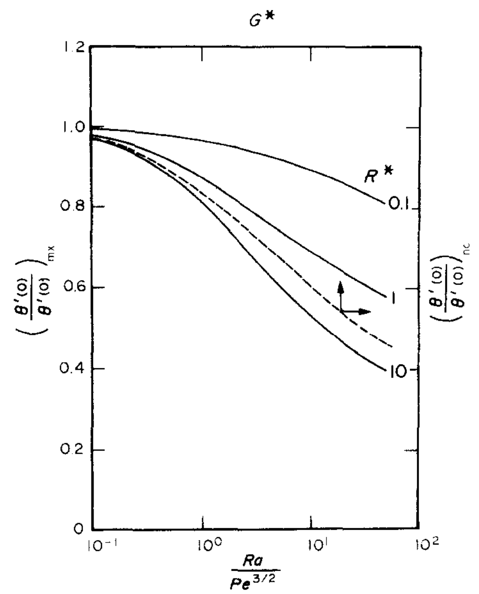

FIG. 6. The ratios of heat transfer coefficient with inertial effects to that with no inertial effects.

by rewriting equation (29) as

$$\frac{Nu}{Pe^{1/2}} = \frac{Nu}{(Ra)^{1/3}}\left[\frac{Ra}{Pe^{3/2}}\right]^{1/3} = \left[\frac{Ra}{Pe^{3/2}}\right]^{1/3}[-\theta'(0)]_{nc} \tag{32}$$

and applying the relation between G^* and R^*

$$G^* = R^*(Ra/Pe^{3/2})^{2/3}. \tag{33}$$

With a given R^* and $Ra/Pe^{3/2}$, G^* can be determined through equation (33). Once G^* is specified, $[-\theta'(0)]_{nc}$ can be solved from equations (14) and (15). Therefore, the free convection asymptote is obtained, from equation (32), for each corresponding R^*.

The results can also be best presented by the ratio of the heat transfer coefficient for the non-Darcy case to that for the Darcy case, which is shown in Fig. 6. It is observed that the inertial term has a pronounced effect on the flow for higher values of G^* and R^*.

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